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On third Poisson structure of KdV equation [1]

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Abstract

The third Poisson structure of KdV equation in terms of canonical "free fields" and reduced WZNW model is discussed. We prove that it is "diagonalized" in the Lagrange variables which were used before in formulation of $2d$ gravity. We propose a quantum path integral for KdV equation based on this representation.

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In this letter we discuss the third Poisson structure of KdV equation and its "diagonalization" in appropriate variables. This is a step towards understanding what is quantum KdV theory – an old question related now to second quantized formulation of string theory, since some particular τ-functions of integrable hierarchies of KdV type can be interpreted as effective action in corresponding string field theory. However, at the moment not really much is known about quantum KdV-type systems. In particular, it is connected to the nontrivial structure of KdV (KP) equation classical phase space.

In this short note we will propose a possible way of formulation what is quantum KdV theory related to its *third* Poisson structure. As in the case of well-known first and second Poisson structures on KdV phase space, the third one is also related to infinite-dimensional Lie algebras and can be realized in terms of canonical free fields [6, 7, 3]. We will base a path-integral formulation on *third* Poisson structure and demonstrate that in certain (Lagrange?) variables its measure and kinetic term acquires a very simple form of "free fields".

1. It was found in [1] and discussed in [2] that in addition to well-known first

$$\{u(x), u(y)\}_1 = \delta'(x-y) \tag{1}$$

and second

$$\{u(x), u(y)\}_2 = (u(x)+u(y))\,\delta'(x-y) + c\delta'''(x-y) \tag{2}$$

Poisson brackets there exists a *non-local* third one

$$\begin{aligned}\{u(x), u(y)\}_3 = \frac{1}{16}\delta'''''(x-y) + \frac{1}{4}\left(u(x)+u(y)\right)\delta'''(x-y) + \frac{1}{8}\left(u'(x)-u'(y)\right)\delta''(x-y) + \\ + \frac{1}{2}\left(u^2(x)+u^2(y)\right)\delta'(x-y) - \frac{1}{4}u'(x)u'(y)\epsilon\,(x-y)\end{aligned} \tag{3}$$

We are going to demonstrate that (3) acquires a very simple form being rewritten in terms of variables coming from two-dimensional WZNW model [3]

$$g = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \tag{4}$$

and $x(\xi) = F^{-1}(\xi)$ [4, 5].

Indeed, starting from the following Poisson structure on a space of variables (4) and implying that [1]

$$\{F(x), F(y)\} = F'(x)F'(y)\epsilon\,(x-y) \tag{5}$$

where

[1] Below we will use $\{A, B\} \equiv \{A, B\}_3$.

$$F(x)=\int_{d\xi}^{x}\exp\alpha\phi\left(\xi\right) \tag{6}$$

with $\phi\left(x\right)$ being common free field so that

$$u\sim\left(\phi'\right)^{2}+\beta\phi''\equiv j^{2}+\beta j' \tag{7}$$

is just a Miura transform with

$$\beta=-\frac{2}{\alpha} \tag{8}$$

or

$$u\sim SchF=\frac{1}{2}\left\{\frac{F'''}{F'}-\frac{3}{2}\frac{F''^{2}}{F'^{2}}\right\} \tag{9}$$

for $\phi\left(x\right)$ one has

$$\{\phi(x),\phi(y)\}=\phi'(x)\phi'(y)\epsilon(x-y)+\frac{1}{\alpha}\left\{\phi'(y)-\phi'(x)\right\}\delta(x-y)-\frac{1}{\alpha^{2}}\delta'(x-y) \tag{10}$$

In such case for the corresponding $U(1)$-current $j(x)=\phi'(x)$ one obtains

$$\begin{aligned}\{j(x),j(y)\}=j'(x)j'(y)\epsilon(x-y)+\left(j(x)j'(y)-j'(x)j(y)\right)\delta(x-y)-\\-j(x)j(y)\delta'(x-y)+\frac{1}{\alpha}\left(j'(x)+j'(y)\right)\delta'(x-y)+\frac{1}{\alpha}\left(j(x)-j(y)\right)\delta''(x-y)+\frac{1}{\alpha^{2}}\delta'''(x-y)\end{aligned} \tag{11}$$

and the Poisson bracket of Sugawara's $u=j^{2}+\beta j'$ coincides with that of (3) provided by (8). Now, a simple remark is that if one considers the *inverse* function

$$x=f(\xi)\equiv F^{-1}(\xi) \tag{12}$$

then it is easy to see that for $x=f(\xi)$ and $y=f(\eta)$

$$\{x,y\}=f'(\xi)f'(\eta)\{\xi,\eta\}=\frac{1}{F'(x)F'(y)}\{F(x),F(y)\}=\epsilon(x-y) \tag{13}$$

or the bracket acquires the most simple form of canonical "free field" bracket for $x(\xi)$

$$\{x(\xi),x(\eta)\}=\epsilon(x(\xi)-x(\eta)) \tag{14}$$

or

$$\{x(\xi),x'(\eta)\}=\delta(\xi-\eta) \tag{15}$$

Finally, let us point out that (5) is a sort of r-matrix bracket

$$\{F(x),F(y)\}=[r(x-y),F(x)\otimes F(y)] \tag{16}$$

with

$$r_{xy} \equiv r(x-y) = \epsilon(x-y)\partial_x \otimes \partial_y$$
$$[r_{xy}, r_{yz}] + [r_{yz}, r_{zx}] + [r_{zx}, r_{xy}] = \delta_{xy}(\epsilon_{yz} + \epsilon_{zx}) + \delta_{yz}(\epsilon_{xy} + \epsilon_{zx}) + \delta_{zx}(\epsilon_{yz} + \epsilon_{xy}) = 0 \tag{17}$$

Eqs. (16) and (17) require some comments. One should consider (17) as an operator acting to *functions* and not to (pseudo)differential operators. Practically it means that

$$[\partial, f(x)] = f'(x) \tag{18}$$

and there are no $f\partial$ terms in the r.h.s. of (18). In terms of algebra $DOP(S^1)$ we are going to discuss below it means that we factor the algebra of differential operators over "differential" terms.

Eq.(16) means that the variables F are "group-like" variables and the corresponding "group" is a Poisson-Lie group, i.e. the third KdV bracket is an example of multiplicative Poisson bracket. A similar r-matrix comes from the standard Lie bialgebra structure for Lie algebra of differential operators $DOP\left(S^1\right)$ of the order ≤ 1 extended in a proper way. Usually a Lie bialgebra structure on $\mathcal{G}$ endows the dual space $\mathcal{G}^*$ with a Lie algebra structure in a such way that the dual map from $\mathcal{G}$ to $\wedge^2\mathcal{G}$ is a 1-cocycle on $\mathcal{G}$ valued in $\wedge^2\mathcal{G}$. As a coalgebra for the Lie algebra $\mathcal{G} = DOP(S^1)$ we have to take the linear space generated by quadratic and linear differentials with a "cocentral" element $\log\partial$. The elements of the coalgebra should be written in the form

$$K\log\partial + \partial^{-1}a + \partial^{-2}b \tag{19}$$

and all terms with higher negative degrees should be truncated in all commutators. The commutation relations in the $DOP\left(S^1\right)$ are the following

$$[f\partial + g, h\partial + e] = \left(fh' - hf'\right)\partial + fe' - hg' + \hat{C}c\left(f\partial + g, h\partial + e\right) \tag{20}$$

where(cf [10])

$$c\left(f\partial + g, h\partial + e\right) = \frac{1}{6}Res\left(fh'''\right) - \frac{1}{2}Res\left(fe'' - hg''\right) - Res\left(ge'\right) \tag{21}$$

Duality is given by the standard Adler-Manin residual trace.

$$\langle f\partial + g + c\hat{C}, K\log\partial + \partial^{-1}a + \partial^{-2}b\rangle = cK + Res(ga) + Res(fb) \tag{22}$$

The only non-trivial commutator in the coalgebra is

$$[\partial^{-1}a, \log\partial] = -\partial^{-2}a' \tag{23}$$

Therefore the co-bracket $\delta : DOP\left(S^1\right) \to \bigwedge^2\left(DOP\left(S^1\right)\right)$ is following

$$\delta(\hat{C}) = \delta(g) = 0$$

$$\delta(f\partial) = f' \otimes \hat{C} - \hat{C} \otimes f' \tag{24}$$

Hence,

$$\langle \delta(f\partial), \partial^{-1}a \otimes \log \partial \rangle = \langle f\partial, [\partial^{-1}a, \log \partial] \rangle = -\langle f\partial, \partial^{-2}a' \rangle = -Res\,(fa') \tag{25}$$

The co-bracket (24) can be written with the help of r-matrix $\delta(x) = [r, \Delta(x)]$ where $x \in \mathcal{G}$ and $r \in \mathcal{G} \otimes \mathcal{G}$. Indeed, if one takes $r_{xy} = 2\epsilon(x-y)$, then

$$\begin{gathered}[\Delta(f\partial), r] = 2[f(x)\partial_x + f(y)\partial_y, \epsilon(x-y)] = \\ = 2\,(f(x) - f(y))\,\delta(x-y) - \hat{C}Res(f(x)\delta'(x-y)) + \hat{C}Res(f(y)\delta'(y-x)) = \\ = \hat{C}\,(f'(x) - f'(y)) = f' \otimes \hat{C} - \hat{C} \otimes f'\end{gathered} \tag{26}$$

Another remark links the expression (5) with integrable systems is that for the bracket (3) the functional $\int u(x)$ is a Hamiltonian for the hamiltonian form of KdV equation. Using (9), we have that

$$H(F) = \int dx Sch\,(F) \tag{27}$$

is a Hamiltonian for Ur-KdV in the coordinates of [3]. The equation is

$$F_t = F''' - \frac{3}{2}\frac{(F'')^2}{F'} \tag{28}$$

and can be rewritten in Hamiltonian form using the Hamiltonian (27) and (non-local) operator

$$\Omega^{-1} = -\frac{1}{2}F'\partial^{-1}F' \tag{29}$$

which is equivalent to the usual KdV written in the hamiltonian form with respect of the third structure (3). These structures were discussed in [9].

The “group” nature of the variables would imply some natural description of the commuting hamiltonians for this equation in terms of central elements of the correspondig coboundary Poisson-Lie “group”.

2. Up to now we have shown that the third Poisson structure is diagonalized in terms of $x(\xi) = F^{-1}(\xi)$ variables having clear sense of one-dimensional or two-dimensional *holomorphic* reparameterizations. These variables are called in hydrodynamics as the Lagrange variables in contrast to the Euler variables $u(x)$, corresponding to the field of velocities of particles of a liquid in a fixed point. On the contrary, the Lagrange variables are nothing but co-ordinates of a fixed ”particle” being exactly $x(t, x_0 = F)$, i.e. coincide with the Polyakov variables [4, 5]. Remarkable enough the same variables arise and make sense in two *a priori* physically different problems.

To formulate the path integral for KdV equation the only additional remark we should make is that the natural measure is also written in terms of Lagrange variables. Indeed, if one takes for example the measure coming from WZNW model [3], then

$$\prod \frac{dF(x)}{F'(x)} = \prod dx(\xi) \tag{30}$$

and the path integral looks as follows

$$Z = \int Dx \exp i\,(S_0 + H) \tag{31}$$

where

$$\begin{aligned} S_0 &= \int_{dt} \theta = \int_{dt} \delta^{-1}\Omega \\ \Omega = \int_{dx\wedge dy} \frac{\delta F(x)}{F'(x)} \wedge \frac{\delta F(y)}{F'(y)} \delta'(x-y) &= \int_{dx} \frac{\delta F(x)}{F'(x)} \frac{d}{dx} \frac{\delta F(x)}{F'(x)} \\ &= \int_{d\xi} \delta x(\xi) \frac{d}{d\xi} \delta x(\xi) \end{aligned} \tag{32}$$

H is a Hamiltonian

$$H(x) = \int_{dtd\xi} x' SchF = -\frac{1}{2} \int_{dtd\xi} \frac{x''(\xi)^2}{x'(\xi)^3} \tag{33}$$

and $Dx = \prod dx$ is *"free" measure* (30). Finally, integrating $\delta\theta = \Omega$ one gets

$$\theta = \frac{1}{2} \int_{d\xi} x'(\xi) \delta x(\xi) \tag{34}$$

so that

$$S_0 = \int_{dtd\xi} \theta = \frac{1}{2} \int_{dtd\xi} \dot{x}x' \tag{35}$$

It is easy to check the consistency of (32) and (35) with the formulas (5), (15) and (29).

3. In this letter we have proved that naively non-local third Poisson structure of the KdV equation acquires a simple and physically clear form if one works in appropriate variables. The choice of convenient variables allows us to write formal path integral in the Hamiltonian form which can be treated as a possible starting point to study quantum KdV theory.

In spite of growing interest to this problem we should mention that it is still an open question. One of the problems is unclear at the moment physical interpretation of the possible result or in other words what should we expect from correlators in quantum KdV theory?

The existence of many different Hamiltonian formulations of classical KdV equation imply a natural question of relations between the path integrals based on different classical formulations. In particular, it is interesting to compare the path integral proposed above with the more common approach to KdV

equation relation to coadjoint orbit quantization of the Virasoro algebra resulting in another path integral [11].

Finally, let us make a remark concerning the "full" path integral. In fact, the exact relation between Euler and Lagrange variables looks as

$$u(x) = b[F(x)]F'(x)^2 + Sch(F) \tag{36}$$

and includes "initial data" b which corresponds to the choice of different orbit [3]. It seems natural to think that the full path integral for KdV theory should include also integration over different orbits, i.e.

$$Z = \int DbDx \exp i\,[S_0(x) + H(b,x)] \tag{37}$$

with some (unknown) measure Db. We are going to return to these questions elsewhere.

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References

[1] F.Magri Journ.Math.Phys., **19** (1978) 1156-1162

[2] B.Enriques, A.Orlov, V.Rubtsov, JETP Lett., October 1993

[3] A.Alekseev, S.Shatashvili Nucl.Phys. **B329** (1989) 719

[4] A.Polyakov Mod.Phys.Lett. **A2** (1987) 893

[5] V.Knizhnik, A.Polyakov, A.Zamolodchikov Mod.Phys.Lett. **A3** (1988) 819

[6] V.Drinfeld, V.Sokolov, Journ.Sov.Math., **30** (1985) 1975-2036

[7] V.Fateev, S.Lukyanov, Int.Journ.Mod.Phys. **A3** (1988) 507; **A7** (1992)

[8] S.Lukyanov, Funct.Anal.appl. **22** (1988) 1

[9] G.Wilson, Phys.Lett. **A132** (1988) 445

[10] E.Arbarello, C.de Koncini, V.Kac, G.Procesi, Comm.Math.Phys. **117** (1988) 1

[11] A.Gorsky, Yad.Fiz. (1991)